arXiv:hep-th/0504148v2 27 Apr 2005

Studying Boojums in $\mathcal{N} = 2$ Theory with Walls and Vortices

R. Auzzi,[a] **M. Shifman,**[a] **and A. Yung**[a,b,c]

[a] *William I. Fine Theoretical Physics Institute, University of Minnesota, Minneapolis, MN 55455, USA*
[b] *Petersburg Nuclear Physics Institute, Gatchina, St. Petersburg 188300, Russia*
[c] *Institute of Theoretical and Experimental Physics, Moscow 117259, Russia*

Abstract

We study 1/2 BPS domain walls, 1/2 BPS flux tubes (strings) and their 1/4 BPS junctions. We consider the simplest example of $\mathcal{N} = 2$ Abelian gauge theory with two charged matter hypermultiplets which contains all of the above-listed extended objects. In particular, we focus on string-wall junctions (boojums) and calculate their energy. It turns out to be logarithmically divergent in the infrared domain. We compute this energy first in the (2+1)-dimensional effective theory on the domain wall and then, as a check, obtain the same result from the point of view of (3+1)-dimensional bulk theory. Next, we study interactions of boojums considering all possible geometries of string-wall junctions and directions of the string magnetic fluxes.

1 Introduction

We are witnessing remarkable advances in the *field-theoretic* studies of branes (domain walls) and strings (flux tubes) [1, 2, 3, 4, 5, 6, 7] with the purpose of revealing and exploring parallels between supersymmetric field theories and string/D-brane theory (for an exhaustive review covering also the issue of confined monopoles see [8]). The unfolding of this program led us well beyond initial anticipations [2, 4, 5].

Supersymmetric gauge theories in $(3 + 1)$ dimensions in the Higgs phase have a rich spectrum of solitons, including domain walls, flux tubes (vortices), confined monopoles and all sorts of junctions. The most convenient set-up exhibiting these solitons, both "elementary" and "composite," which emerged in the recent years, is provided by U(N) Yang–Mills theory with extended $\mathcal{N} = 2$ supersymmetry (SUSY) and the Fayet–Iliopoulos term for the $U(1)$ factor (see Refs. [9, 10, 11, 12, 5, 13, 14, 15, 16, 17]). The matter sector includes $N_f \geq N$ hypermultiplets.

Among other intriguing features revealed *en route*, let us mention just a few results: (i) localization of an effective U(1) gauge theory on an elementary domain wall; (ii) demonstration that Abelian vortices can end on walls and act as massive magnetic sources for the effective U(1) gauge theory localized on the wall; (iii) localization of non-Abelian gauge fields on composite walls, with non-Abelian flux tubes ending on them, etc.

Besides giving an extensive review of the subject, Ref. [8] presents a new result regarding the wall-string junctions, *boojums*[1]. Our task is to further elaborate on this issue. In this paper we discuss aspects of the binding energy of flux tubes ending on walls in the "minimal" setting of $\mathcal{N} = 2$ SQED with two flavor hypermultiplets. We also dwell on multi-boojums. We find that the negative finite binding energy observed in [8] is in fact ill-defined

---

[1]The word *boojum* comes from the Lewis Carroll's children's book *Hunting of the Snark*. Apparently, it is fun to hunt a snark, but if the snark turns out to be a boojum, you are in trouble! Condensed matter physicists adopted the name in the 1970s to describe solitonic objects of the wall-string junction type in helium-3 and other systems. Also: *boojum tree* (Idria Columnaris) is a tree endemic to the Baja California peninsula of Mexico. The boojum tree is one of the strangest plants imaginable. For most of the year it is leafless and looks like a giant upturned turnip. Its common name was coined by the plant explorer Godfrey Sykes, who found it in 1922 and said "It must be a boojum!" In saying this, Sykes was referring to Lewis Carrol's book. The Spanish common name for this tree is Cirio, referring to its candle-like appearance. Supersymmetric wall-string junctions were referred to as *boojums* in Ref. [8].

and shadowed by certain logarithmic contributions in single boojums, while it remains well-defined for certain multi-boojums. In addition we consider forces between the flux tubes in the multi-boojum configurations.

The organization of the paper is as follows. The results of Ref. [4] relevant for the subsequent consideration are summarized in Sect. 2. Section 3 presents a calculation of energy of the wall-vortex system. In Sect. 4 various two-boojum configurations, in which two vortices end on the same wall, are discussed. In Sect. 5 we formulate a slightly modified approach to define a boojum self-energy, while Sect. 6 contains our conclusions.

2 The bulk, the wall & the vortex

The bulk theory is $\mathcal{N} = 2$ SQED with 2 flavors and a Fayet–Iliopoulos term. The bosonic part of the Lagrangian is

$$
S = \int d^4x \left\{ \frac{1}{4g^2} F_{\mu\nu}^2 + \frac{1}{g^2} |\partial_\mu a|^2 + \bar{\nabla}_\mu \bar{q}_A \nabla_\mu q^A + \bar{\nabla}_\mu \tilde{q}_A \nabla_\mu \bar{\tilde{q}}^A \right.
$$

$$
\left. + \frac{g^2}{8} (|q^A|^2 - |\tilde{q}_A|^2 - \xi) + \frac{g^2}{2} |\tilde{q}_A q^A|^2 + \frac{1}{2} (|q^A|^2 + |\tilde{q}^A|^2)|a + \sqrt{2} m_A|^2 \right\},
$$

$$(1)$$

where

$$
\nabla_\mu = \partial_\mu - \frac{i}{2} A_\mu
$$

and

$$
\bar{\nabla}_\mu = \partial_\mu + \frac{i}{2} A_\mu .
$$

The index $A = 1, 2$ is the flavor index; the mass parameters m_1, m_2 are assumed to be real. We will work in the limit

$$
\Delta m = m_1 - m_2 \gg g\sqrt{\xi} .
$$

There are two vacua in this theory: in the first vacuum

$$
a = -\sqrt{2} m_1, \qquad q_1 = \sqrt{\xi}, \qquad q_2 = 0 , \tag{2}
$$

and in the second one

$$
a = -\sqrt{2} m_2, \qquad q_1 = 0, \qquad q_2 = \sqrt{\xi} . \tag{3}
$$

The vacuum expectation value (VEV) of field $\tilde{q}$ vanishes in both vacua. Hereafter we will stick to the *ansatz* $\tilde{q} = 0$.

A BPS domain wall interpolating between the two vacua of our bulk theory was explicitly constructed in Ref. [4]. Assuming that all fields depend only on the coordinate $z = x_3$, it is possible to write the energy in the Bogomolny form [18],

$$\int d^3x \, \mathcal{H} = \int dx_3 \left\{ \left| \nabla_3 q^A \pm \frac{1}{\sqrt{2}} q^A (a + \sqrt{2} m_A) \right|^2 \right.$$

$$+ \left. \left| \frac{1}{g} \partial_3 a \pm \frac{g}{2\sqrt{2}} \left(|q^a|^2 - \xi \right) \right|^2 \pm \frac{1}{\sqrt{2}} \xi \partial_3 a \right\}. \qquad (4)$$

Putting the first two terms above to zero gives us the BPS equations for the wall. Assuming that $\Delta m > 0$ we choose the upper sign in (4). The tension is given by the total derivative term, (the last one in Eq. (4)) which can be identified as the $(1, 0)$ central charge of the supersymmetry algebra,

$$T_{\rm w} = \xi \, \Delta m \,. \qquad (5)$$

The wall solution has a three-layer structure (see Fig. 1): in the two outer layers (which have width $O((g\sqrt{\xi})^{-1})$) the squark fields drop to zero exponentially; in the inner layer the field a interpolates between its two vacuum values. The thickness of this inner layer is given by

$$R = \frac{4\Delta m}{g^2 \xi}. \qquad (6)$$

This wall is an $1/2$ BPS solution of the four-dimensional action. In other words, the soliton breaks 4 of the 8 supersymmetry generators of $\mathcal{N} = 2$ bulk theory. The moduli space is described by two bosonic coordinates: one of these coordinates is associated with the wall translation; the other one is an U(1) compact parameter. It is possible to promote these moduli to fields depending on the wall coordinates x_n ($n = 0, 1, 2$). A world-volume theory for the moduli fields on the wall can be written. The bosonic part of the world-volume action is

$$S_{2+1} = \int d^3x \left(\frac{T_{\rm w}}{2} (\partial_n \zeta)^2 + \frac{1}{4e_{2+1}^2} (F_{mn}^{(2+1)})^2 \right), \qquad (7)$$
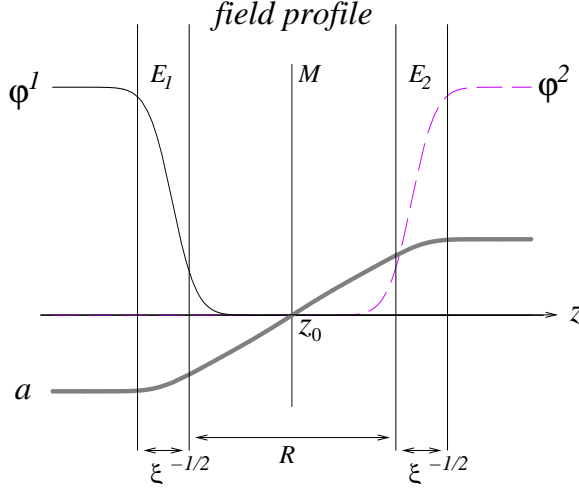
3

Figure 1: Internal structure of the domain wall: two edges (domains $E_{1,2}$) of the width $\sim \xi^{-1/2}$ are separated by a broad middle band (domain M) of the width $R \sim \Delta m/(g^2 \xi)$.

where ζ describes the translational mode while e_{2+1}^2 is the coupling constant of the effective U(1) on the wall,

$$e_{2+1}^2 = 4\pi^2 \frac{\xi}{\Delta m} \,. \tag{8}$$

The fermion content of the world-volume theory is given by two three-dimensional Majorana spinors, as is requited by $\mathcal{N} = 2$ in three dimensions. The full world-volume theory is U(1) Yang–Mills theory in (2+1) dimensions, with four supercharges. The Lagrangian and the corresponding superalgebra can be obtained by reducing four-dimensional $\mathcal{N} = 1$ SQED (with no matter) to three dimensions.

Our theory (1) is an Abelian theory so it does not have monopoles. However we can start from non-Abelian $\mathcal{N} = 2$ theory with the gauge group SU(2) broken down to U(1) theory (1) by the condensation of the adjoint scalar field (whose third component is a, see (2) and (3)). This non-Abelian theory have 't Hooft-Polyakov monopoles. In the low-energy theory (1) they become heavy external magnetic charges with mass of order of m/g^2. As soon as electrically charged states condense in both vacua (2) and (3) these monopoles are in the confinement phase in both vacua.

In fact, in each of the two vacua of the bulk theory magnetic charges are confined by the Abrikosov–Nielsen–Olesen (ANO) strings. In the vacuum in

4

which $a = \sqrt{2}m_1$ we can use the *ansatz* $q_2 = 0$ and write the following BPS equations (see [18, 19]):

$$F_3^* - \frac{g^2}{2}(|q^1|^2 - \xi) = 0, \quad (\nabla_1 - i\nabla_2)q^1 = 0, \tag{9}$$

where

$$F_i^* = 1/2\varepsilon_{ijk}F_{jk}, \qquad i, j, k = 1, 2, 3.$$

Analogous equations can be written for the flux tube in the other vacuum, $a = \sqrt{2}m_2$. These objects are half-critical, much in the same way as the domain walls above.

The magnetic flux of the minimal-winding flux tube is 4π, while its tension is given by the $(1/2, 1/2)$ central charge [20],

$$T_s = 2\pi\xi. \tag{10}$$

The thickness of the tube is of the order of

$$r_0 = (g^2 \xi)^{-1/2}. \tag{11}$$

3   The boojum (wall–flux-tube junction)

Let us consider a flux tube ending on a domain wall; this configuration has been studied in Refs. [4, 5, 15, 8]. Our primary goal is examination of the binding energy.

The magnetic flux carried by the monopole is unconfined inside the domain wall; so the vortex-wall junction behaves as a charge in the Coulomb phase of the wall world-volume theory. Let us consider a vortex oriented in the z direction ending on a wall oriented in the x_1, x_2 directions; let the string extend at $z > 0$ and let the magnetic flux be oriented in the positive z direction. BPS first order equations can be written for the composite soliton,

$$F_1^* - iF_2^* + \sqrt{2}(\partial_1 - i\partial_2)a = 0,$$

$$F_3^* + \frac{g^2}{2}(|q^A|^2 - \xi) + \sqrt{2}\partial_3 a = 0,$$

$$\nabla_3 q^A = -\frac{1}{\sqrt{2}}q^A(a + \sqrt{2}m_A),$$

$$(\nabla_1 - i\nabla_2)q^A = 0. \tag{12}$$

5

These equations generalize both the 1/2 BPS wall and the flux-tube equations and can be used for determination of the general 1/4 BPS boojum soliton, (i.e. a flux tube ending on a wall).
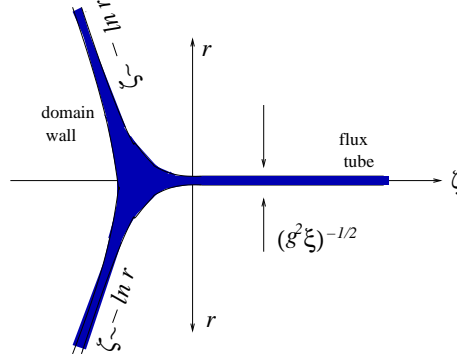

Figure 2: Bending of the wall due to the string-wall junction. The flux tube extends to the right infinity. The wall profile is logarithmic at transverse distances larger than $g^{-1}\xi^{-1/2}$ from the string axis. At smaller distances the adiabatic approximation fails.

At large distance r from the string junction the wall is logarithmically bent due to the fact that the vortex pulls it (see Fig. 2),

$$z = -\frac{1}{\Delta m} \ln r + \text{const}.$$
(13)

First of all, let us estimate the binding energy of the boojum from the standpoint of the world-volume theory. At large distances r from the string-wall junction the fields in the $(2+1)$ effective theory are given by the following expressions:

$$F_{0i}^{2+1} = \frac{e_{2+1}^2}{2\pi} \frac{x_i}{r^2}, \qquad \zeta = -\frac{1}{\Delta m} \ln r + \text{const},$$
(14)

where $i = 1, 2$. There are two contributions to the energy of this field configuration. The first contribution to the energy localized at $r_0 < r < r_f$ is due to the gauge field,

$$E_{(2+1)}^G = \int_{r_0}^{r_f} \frac{1}{2e_{2+1}^2} (F_{0i})^2 \, 2\pi r \, dr$$

$$= \frac{\pi \xi}{\Delta m} \int_{r_0}^{r_f} \frac{dr}{r} = \frac{\pi \xi}{\Delta m} \ln \frac{r_f}{r_0}.$$
(15)

6

The second contribution, due to the ζ field is,

$$
\begin{aligned}
E_{(2+1)}^{H} &= \int_{r_0}^{r_f} \frac{T_{\mathrm{w}}}{2} (\partial_r \zeta)^2 \, 2\pi r \, dr \\[2mm]
&= \frac{\pi \xi}{\Delta m} \ln \frac{r_f}{r_0} \, .
\end{aligned}
\tag{16}
$$

These two contributions are both logarithmically divergent at $r_f \to \infty$, this is an infrared (IR) divergency. Its occurrence is an obvious feature of the charged objects in $(2+1)$ dimensions due to the fact that the fields vanish too slowly at infinity to have a finite energy. There is no divergence in the ultraviolet (UV) domain, at r_0. In fact, the value of r_0 is given by the thickness of the flux tube, which is of the order of $r_0 \sim (g\sqrt{\xi})^{-1}$.

The above two contributions are equal (with the logarithmic accuracy), even though their physical interpretation is different. The total "additional" energy is

$$
E^{G+H} = \frac{2\pi \xi}{\Delta m} \ln \frac{r_f}{r_0} \, .
\tag{17}
$$

The gauge field contribution can be interpreted in the bulk theory as the energy carried by the magnetic field. The corresponding energy density is

$$
\rho^{G} = \frac{1}{2g^2} \vec{B}^2
\tag{18}
$$

inside the domain wall of thickness R. This can be seen from the following $(3+1)$ bulk calculation:

$$
E^{G} = \int_{r_0}^{r_f} \frac{1}{2g^2} |\vec{B}|^2 \, R \, 2\pi r \, dr = \frac{\pi \xi}{\Delta m} \ln \frac{r_f}{r_0} \, .
\tag{19}
$$

The ζ field contribution can be interpreted in $(3+1)$ dimensions as due to the fact that the domain wall bends and, therefore, its area S changes, $\Delta S = \pi (\Delta m)^{-2} \ln(r_f/r_0)$, and

$$
\begin{aligned}
E^{H} &= T_{\mathrm{w}} \Delta S \\[2mm]
&= T_{\mathrm{w}} \int_{r_0}^{r_f} \left(\sqrt{1 + \frac{1}{(\Delta m)^2 r^2}} - 1 \right) 2\pi r \, dr \approx \frac{\pi \xi}{\Delta m} \ln \frac{r_f}{r_0} \, .
\end{aligned}
\tag{20}
$$

7

Equation (20) implies that the "Higgs-induced" logarithmic term in the energy of the boojum configuration (its geometry is depicted in Fig. 3) can be absorbed in the actual area of the curved wall,

$$S = \pi\, r_f^2 + \Delta S\,. \qquad (21)$$

One then measures the string length from the origin, see the parameter ℓ in Fig. 3. This interpretation seems natural.
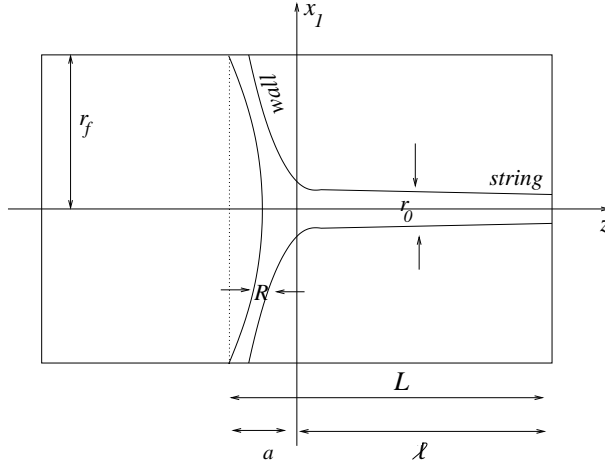

Figure 3: Geometry of the boojum configuration

Our composite soliton is quarter-critical. Therefore, it is possible to use the central charges approach to calculate its energy. There are three types of central charges in $\mathcal{N} = 2$ theories: the $(1,0)$ charge, which can be saturated by domain walls; the $(1/2, 1/2)$ charge, which can be saturated by vortices; the $(\mathcal{N} = 2)$ Lorentz-scalar charge, which can be saturated by monopoles. When a BPS composite soliton such as a boojum is treated, we have to consider all these contributions to the total energy of the object that we are considering. Assuming that BPS equations are satisfied, the energy of the composite soliton is given by

$$\int d^3x\,\mathcal{H} = \int d^3x \left\{ \frac{1}{\sqrt{2}}\xi\partial_3 a + \xi\frac{F_3^*}{2} - \frac{1}{g^2}\sqrt{2}\,\partial_\alpha\left(aF_\alpha^*\right) \right\}, \qquad (22)$$

where the three terms are respectively the $(1,0)$, the $(1/2, 1/2)$ and the Lorentz scalar central charges (the latter is $\mathcal{N} = 2$). Let us consider the

energy inside the domain $x_1^2 + x_2^2 < r_f^2$ and of length ℓ in the z direction. The distance ℓ is measured starting from the intersection of the tube with the wall surface. As is clear from Fig. 3,

$$L = a + \ell \qquad (23)$$

where

$$a = \frac{1}{\Delta m} \ln \frac{r_f}{r_0} . \qquad (24)$$

The first term in (22) gives, in the first approximation, the wall tension T_w times the area πr_f^2. The second term can be written as follows:

$$\int d^3x \, \xi \, \frac{F_3^*}{2} = \frac{\xi}{2} \left(\int dx_1 dx_2 F_3^* \right) L \approx 2\pi \, \xi \, L . \qquad (25)$$

reflecting the fact that the magnetic flux through different sections in the x_1, x_2 plane is conserved and equal to 4π. The expression for L is given in Eqs. (23) and (24). The estimate above is good at order $O(\Delta m/g^2)$ because the coordinate z corresponding to the left edge has uncertainty of the order of the wall thickness R: the slices in the x_1, x_2 plane with radii close to r_f fail to capture all the magnetic flux.

The energy in Eq. (25) contains both the energy due to the "actual" string tension (which is $2\pi\xi\ell$) and the energies $E^G + E^H$ (i.e. the surface bending plus the world-volume charge energy, which is equal $2\pi\xi a$, in total). Now, we will derive this result from a direct calculation at small z. We have

$$E_{0<z<a} = \frac{\xi}{2} \int_0^a dz \int dx_1 dx_2 F_3^* = \frac{\xi}{2} 2\pi \int r dr \int_{\zeta-R/2}^{\zeta+R/2} dz |F^*||\partial_r \zeta|, \qquad (26)$$

where F^* is the vector of the magnetic flux inside the wall, while the factor $\partial_r \zeta$ produces its z-component, see Fig. 4. The magnetic flux inside the string is directed along the z-axis and is equal to 4π. This flux is spread out inside the wall and directed along the wall surface to the point of the wall-string junction, see Fig. 4. We have inside the wall

$$|F^*| = \frac{2}{R} \frac{1}{r}, \qquad (27)$$

see [4] for details. Substituting this into Eq. (26) and using (13) we finally get

$$E_{0<z<a} = \frac{\xi}{2} \int_0^a dz \int dx_1 dx_2 F_3^* = \frac{2\pi\xi}{\Delta m} \ln \frac{r_f}{r_0} = 2\pi \xi \, a. \qquad (28)$$
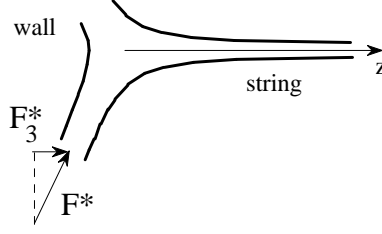
Figure 4: Direction of the magnetic flux inside the wall.

There is an ambiguity in defining the distance L due to the fact that the wall has a finite thickness of order

$$\frac{\Delta m}{g^2 \xi}.$$

This is reflected in an uncertainty in this estimate of the energy of the order of

$$O\left(\frac{\Delta m}{g^2}\right).$$

The last term of (22) gives a finite negative contribution

$$-\frac{4\pi}{g^2}\Delta m$$

discussed in Ref. [8]; this is of the same order as the uncertainty due to ambiguity in the definition of the integration domain of the Hamiltonian. The total energy of the soliton, which is defined up to a quantity of the order of

$$O\left(\frac{\Delta m}{g^2}\right),$$

is given by the following expression:

$$
\begin{aligned}
E &= \int d^3 x \mathcal{H} \approx T_{\mathrm{w}} \pi r_f^2 + T_{\mathrm{s}} L \\
&= T_{\mathrm{w}} \pi r_f^2 + \frac{2\pi\xi}{\Delta m} \ln \frac{r_f}{r_0} + T_{\mathrm{s}} \ell.
\end{aligned}
\tag{29}
$$

Let us introduce S_0, the area of a would-be unbent wall,

$$S_0 = \pi r_f^2. \tag{30}$$

10

Equation (29) implies that the logarithmic terms in the energy of the boojum configuration can be absorbed in various ways. For instance, in terms of the actual area S of the curved wall,

$$S = S_0 + \Delta S$$

and measuring the string length from the origin, (i.e. length= ℓ in Fig. 3) we have

$$
\begin{aligned}
E &= T_{\mathrm{w}}S_0 + E^H + E^G + T_{\mathrm{s}}\ell \\
&= T_{\mathrm{w}}S + E^G + T_{\mathrm{s}}\ell,
\end{aligned}
\tag{31}
$$

where E^G is calculated in Eq. (19).

On the other hand, in terms of an imaginary string length $L = a + \ell$ the total energy is given by

$$E = T_{\mathrm{w}}S_0 + T_{\mathrm{s}}L \,. \tag{32}$$

We see that the energy gain in the wall world-volume equals to the energy loss due to contraction of L. We will use this fact in Sect. 4.

In both cases there is an intrinsic uncertainty in defining the above geometric parameters. It is clear that the curved wall area S can be defined only up to $\sim r_0^2$ which leads to the energy uncertainty

$$\sim T_{\mathrm{w}}r_0^2 \sim \Delta m/g^2 \,.$$

If S is fixed to be πr_f^2 (i.e. the bending is excluded) and is well-defined, the uncertainty in L is of the order of R implying the energy uncertainty

$$\sim T_{\mathrm{s}}R \sim \Delta m/g^2 \,.$$

Both derivations perfectly match. The resulting uncertainty is of the same order as the negative binding term of Sakai and Tong.

4 String interaction in multi-boojums

Let us first of all consider the situation in which the two vortices are on opposite sides of the wall and carry the same positive U(1) flux in the z direction (see Fig. 5). The distance between the two strings in the x_1, x_2 plane is denoted by d. The two vortices pull the wall from different directions
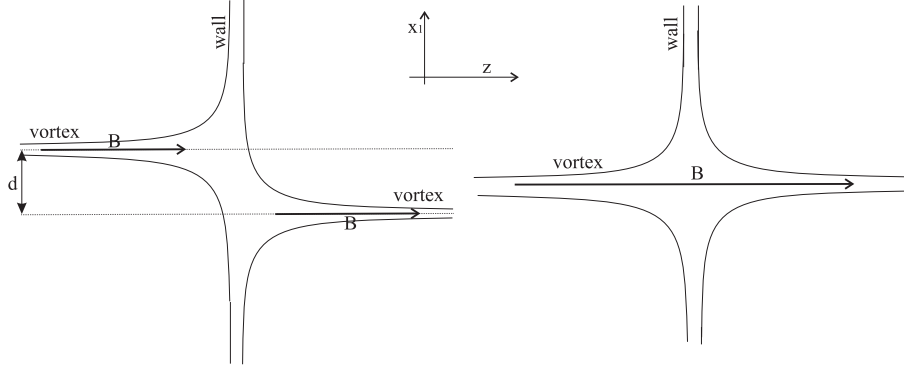
Figure 5: Vortices on opposite sides of the wall which carry the same positive U(1) flux in the z direction. This is a deformation with the same energy of the configuration of a vortex crossing the wall.

and so there is no asymptotic bending at a distance much larger than d. The two wall-vortex junctions can interact throughout the gauge and Higgs fields; it is easy to calculate these interactions in the wall world-volume theory in the regime of large d.

The gauge flux produced by two junctions has the form

$$F_{0i}^{2+1} = \frac{e_{2+1}^2}{2\pi} \left[\frac{(x-x_1)_i}{|x-x_1|^2} - \frac{(x-x_2)_i}{|x-x_2|^2} \right],$$ (33)

where $i = 1, 2$ and x_1 and x_2 are the positions of two junctions on the wall, $d = x_1 - x_2$. Note the opposite signs of two terms here. This is because there is no gauge flux inside the wall at large r in the configuration shown in Fig. 5. In other words the electric charges of two junctions are opposite. Substituting this flux in the world volume action (7) we get the energy of two junctions induced by the gauge interaction

$$V^G = \frac{2\pi\xi}{\Delta m} \ln \frac{d}{r_0}.$$ (34)

This potential gives an attractive force. Note the cancellation of IR-divergent "self-energy" part. This happens because there is no flux at infinity in this configuration.

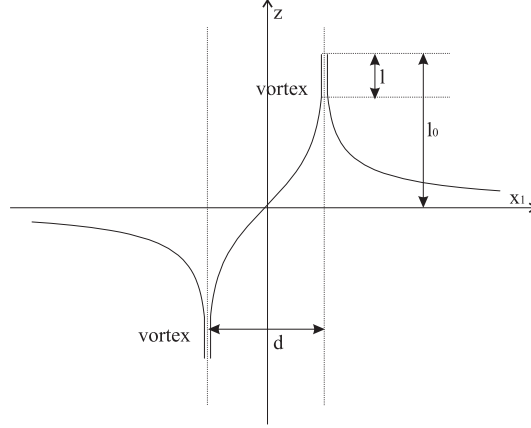Now consider the potential induced by Higgs interactions. The wall bend-

12

Figure 6: Geometry of a two-boojums configuration with the vortices on the different sides of the wall at large distance d. The wall thickness is considered zero in this picture. The bending of the wall is given by $z = \frac{1}{\Delta m} \ln \left| \frac{x_1 + d/2}{x_1 - d/2} \right|$.

ing produced by two junctions has the form (see Fig. 6):

$$\partial_i \zeta = \frac{1}{\Delta m} \frac{(x - x_1)_i}{|x - x_1|^2} - \frac{1}{\Delta m} \frac{(x - x_2)_i}{|x - x_2|^2}. \tag{35}$$

Two terms here have different signs again because there is no wall bending at large r for the configuration in Fig. 5. Substituting this in (7) we get

$$V^H = \frac{2\pi \xi}{\Delta m} \ln \frac{d}{r_0}. \tag{36}$$

This potential is also attractive. Note again the cancellation of the IR-divergent part (note that it is possible to provide a bulk interpretation to this contribution as due to a stretch of one flux tube in the presence of the wall logarithmic deformation caused by the other).

So far we considered interaction induced by the wall bending and the presence of the flux inside the wall. There is also another effect which we have to take into account: the length of both string changes as we change the distance d between junctions. Let us denote by l_0 the length of each string at $d = 0$ when the wall is not bent. Then the actual length of the string as a function of distance d is

$$l = l_0 - \frac{1}{\Delta m} \ln \frac{d}{r_0}, \tag{37}$$
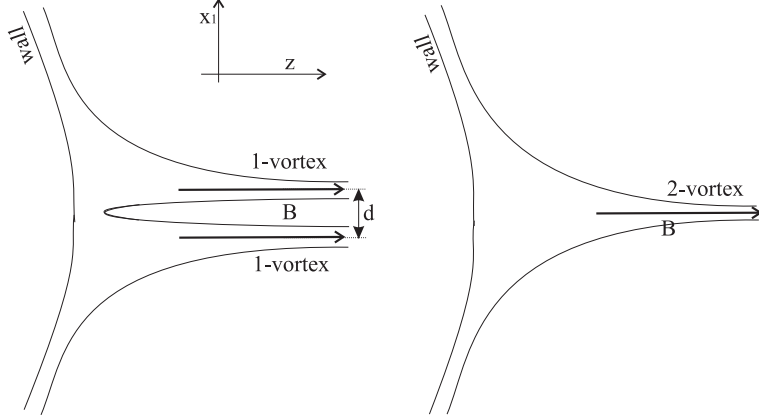
13

Figure 7: Two elementary vortices on the same side of the wall which carry the same positive U(1) flux in the z direction. This is a deformation of the configuration of a two-winding vortex ending on a wall.

see (13), (35) and Fig. 6. Note that ambiguity in defining the distance between the junctions results in the ambiguity in the argument of the logarithm. The coefficient in front of the logarithm is determined unambiguously. Taking into account that the string tension is equal to $2\pi\xi$ and we have two strings we get the following repulsive potential

$$V^S = -\frac{4\pi\xi}{\Delta m}\ln\frac{d}{r_0}. \tag{38}$$

Assembling all three contributions in (34), (36) and (38) we see that string junctions do not interact,

$$V = V^G + V^H + V^S = 0. \tag{39}$$

Thus, there is no effective force in the wall world-volume between the two junctions.

This calculation suggests that there is no force between the two vortices not only in the logarithmic regime, but also in the small-distance regime (where the logarithmic approximation that we exploit fails). We expect that d is a modulus of the composite soliton. It should be possible then to realize this configuration for arbitrary d as an 1/4-BPS solution of equations (12).

This type of solution allows us to describe the deformation of an ANO vortex traversing the domain wall as two vortices ending on the wall at separate points. The situation is similar to what happens in string theory to
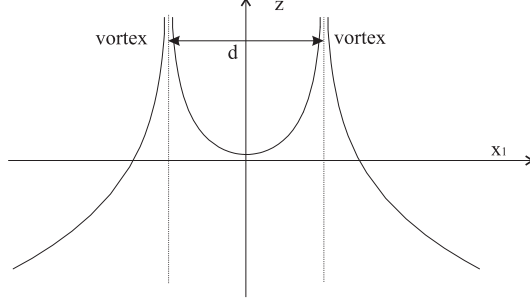
14

Figure 8: Geometry of a two-boojums configuration with both the vortices on the same side of the wall at large distance d. The wall thickness is considered zero in this picture. The bending of the wall is given by $z = \frac{1}{\Delta m} \ln \left| \frac{r_0^2}{(x_1 - d/2)(x_1 + d/2)} \right|$.

a string traversing a brane (see, for example, [21] for a discussion of the problem using the Born-Infeld action).

The total energy of this composite BPS object is given by the sum of the three central charges (see Eq. (22)). Let us consider the energy inside a large cylinder

$$x_1^2 + x_2^2 < r_f^2$$

of length l in the z direction; this domain is chosen in such a way that it contains both the vortex junctions and both flat faces of the cylinder are at large distances from the wall plane. The result for the energy is

$$\int d^3x \mathcal{H} = T_{\mathrm{w}} \pi r_f^2 + T_{\mathrm{s}} l - \frac{8\pi}{g^2} \Delta m \,, \qquad (40)$$

($l = 2l_0$). The constant term due to the Lorentz-scalar central charge (which is discussed in Ref. [8]) is *well-defined in this expression*. Note that ambiguity in the arguments of the logarithms in Eqs. (34), (36) and (38) produces an effect of the order of $\xi/\Delta m$, which is much smaller than the last term in the right-hand side of Eq. (40).

Our theory is Abelian and, as such, does not have monopole solutions. However, if we started from a non-Abelian $\mathcal{N} = 2$ theory with the gauge group $U(2)$, we would have confined monopoles attached to the end points of the flux tubes, both in the first and the second vacua (see Refs. [12, 13]). The constant term in Eq. (40) can be interpreted then as the mass difference between these two monopoles (see also Ref. [8] for a qualitative description
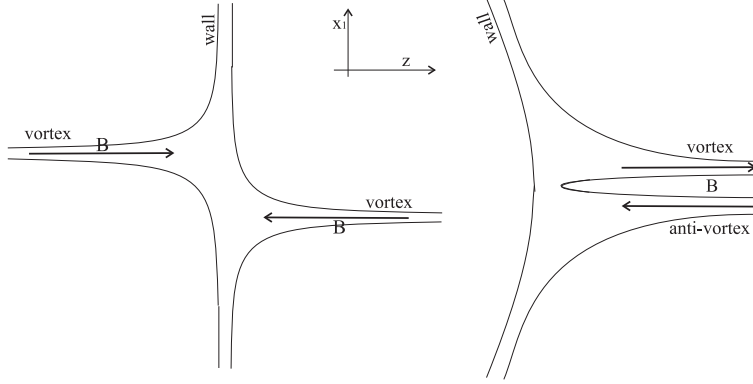
15

Figure 9: No BPS solution exists for these configurations; in the case on the left the force between the junctions is repulsive while in the case on the right the force is attractive.

of a monopole passing through a domain wall with two collinear strings on each side).

There is another configuration in which the world-volume Coulomb and Higgs repulsion are exactly canceled by the "string length" attraction: the one in which the two vortices are both on the same side of the wall with the same magnetic flux orientation, see Figs. 7 and 8. This is consistent with the behavior of the vortices in the bulk: far from the wall there is no force between them because they are relatively BPS.

In this case, as previously, we expect the composite soliton to be 1/4 BPS saturated, and the relative distance d to be a modulus. The computation of energy of this system is completely similar to the one for a single string ending on a wall; in particular, there is a $\ln r$ divergence in the binding energy due to the $(1/2, 1/2)$ central charge.

On the other hand, the situation is different if the vortices have the opposite magnetic fluxes in the z direction, see Fig. 9. If the vortices stay on different sides of the wall, the Higgs interaction between their junctions is attractive while the electric interaction is repulsive. These effects cancel each other. The resulting total potential is due to the "string length" repulsion,

$$V = -\frac{4\pi\xi}{\Delta m} \ln \frac{d}{r_0} \,. \tag{41}$$

There is no chance to obtain a BPS configuration.

A similar thing happens when two vortices are on the same side of the

16

wall: the total potential is due to the "string length" attraction,

$$V = \frac{4\pi\xi}{\Delta m} \ln \frac{d}{r_0} \,.$$
(42)

The vortex and the anti-vortex will attract and annihilate each other; the interaction will be stronger on the wall world-volume where the theory is in the Coulomb phase, and will be exponentially suppressed in the bulk by distance between the vortices.

5   Boojum self-energy: the second try [2]

In Sect. 3 we calculated the self-energy of a single boojum. We found this energy to be logarithmically divergent in the IR limit $r_f \to \infty$, see Eq. (17). The attempt to isolate a finite binding energy of the boojum of the order of $\Delta m/g^2$ failed because of a definition-dependence of the parameters ℓ and S.

On the other hand, in Sect. 4 it was explained that in a certain two-boojum configuration all logarithmic terms cancel, see Eq. (39), relevant geometric parameters l and $S = \pi r_f^2$ become well-defined and, as a result, the two-boojum energy contains a well-defined [8] finite term $-8\pi\Delta m/g^2$, see Eq. (40).

The question arises as to whether one can take advantage of this situation to unambiguously determine the single-boojum binding energy. To this end one can try to use the following natural strategy.

Consider the two-boojum configuration depicted in Fig. 5. We will refer to this configuration as to boojum-antiboojum system because it is characterized by the absence of both, the wall bending and flux at $r \to \infty$ with d fixed. As was mentioned, all logarithms cancel, but a finite negative binding energy *must* exists, as is obvious *a priori* from comparison of two graphs presented in Fig. 10.

The very fact of negative binding energies for junctions is not surprising and is wide-spread. For instance, a negative tension for the multi-domain wall junction in a Wess-Zumino model was found in Ref. [22]. The reason underlying this phenomenon was explained in [23]: the local energy density is diluted inside the junction. Figure 11 borrowed from Ref. [23] illustrates

---

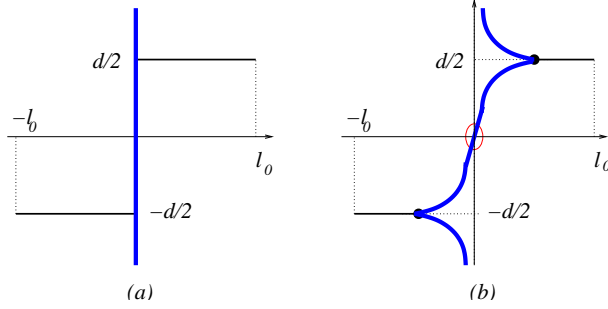[2]This section was written as a result of discussions with David Tong to whom we are deeply grateful.

Figure 10: A hypothetical (a) and actual (b) boojum-antiboojum configuration. In the logarithmic approximation both have the same energies. Since the *bona fide* solution is presented by (b), it must have a negative finite binding energy.

this statement. It clearly shows that the energy density inside the junction is lower than that inside the walls (why this is the case is also understood).
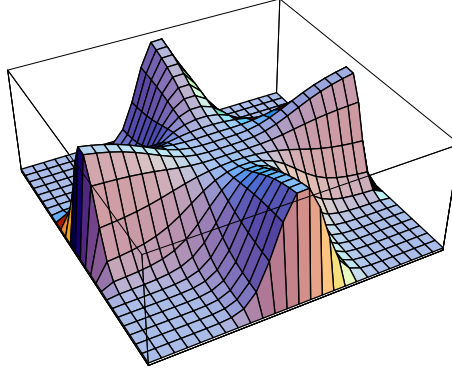

Figure 11: Energy density of the domain wall junction

Let us turn to the boojum-antiboojum system and start from vanishing separation, $d = 0$ see Fig. 5. The plot on the right corresponds to the string crossing the wall with no long range interactions. There is no doubt that the binding energy of this configuration, $-8\pi\Delta m/g^2$, is localized inside the bulge at the origin, where the energy density is diluted compared to that outside the bulge. (It is not difficult to see that the energy density inside the wall and both flux tubes is of the order of $g^2\xi^2$.)

Now, let us gradually increase d passing to the IR limit $d \gg R$, r_0. There

are two logical possibilities. (i) The overall binding energy $-8\pi\Delta m/g^2$, which remains intact, splits into two domains of energy-density deficit, each localized at the position of the corresponding junction.[3] Then, this would provide us with a sensible definition/determination of a finite binding energy of the isolated boojum in the form

$$E = -\frac{8\pi}{g^2}\Delta m \cdot \left(\frac{1}{2}\right) \tag{43}$$

per boojum, in full accordance with Ref. [8]. (ii) The total binding energy $-8\pi\Delta m/g^2$ is localized near the junctions *and elsewhere*. On physical grounds, it cannot spread evenly between the junctions. We would have captured such a spread in the logarithmic terms. On the other hand, it is intuitively clear that the *third localized domain* of energy-density deficit might emerge near the origin (see the red oval in Fig. 10b). This is a transition domain between the positive and negative logarithmic bending; in this domain the wall is bent to a lesser extent than outside this domain. Thus, the energy due to the Higgs field is expected to be lower here than outside this domain. On the other hand, the energy due to the gauge field flux may or may not be higher. One can estimate the order of magnitude of possible energy deficit near the origin. Using the fact that the local energy density $\sim g^2\xi^2$, in conjunction with Eqs. (6) and (11), we arrive at the energy deficit $\sim \Delta m/g^2$, i.e. of the same order of magnitude as in Eq. (43). If so, then the binding energy of the isolated boojum cannot be defined in this way, i.e. as half of the binding energy of the boojum-antiboojum configuration at large d.

Needless to say, the above intuitive observation of a central domain of an energy deficit can only be firmly confirmed by a direct numerical solution for the boojum-antiboojum configuration analogous to that carried out in Ref. [23]. Then we could analyze the local energy density point-by-point inside the boojum-antiboojum configuration and observe (or rule out) an energy-density deficit in the central domain.

6 Conclusions

In the present paper we studied 1/4-BPS boojums in the simplest possible model, $\mathcal{N} = 2$ QED. We calculated the energy of an isolated boojum and

---

[3]This option is advocated by David Tong.

found that it is logarithmically divergent in the infrared domain. This divergence appears because the string end represents an electric charge in the effective theory on the wall, and moreover, the string produces a logarithmic bending of the wall. We compute this energy in three different ways. First, we calculate it from the point of view of (2+1)-dimensional effective theory on the wall. Then we rederive the same result in the bulk theory: it comes from an effective increase of the wall surface due to its bending and the string magnetic flux spread inside the wall. Finally we confirm this result studying contributions to the central charges (22) of different BPS objects.

There are uncertainties in defining the area of the wall and the length of the string. These uncertainties produce ambiguous finite contributions to the boojum energy, of the order of the monopole mass $\sim m/g^2$. They are of the same order as the result for the boojum energy obtained in Ref. [8].

Next we study multi-boojums. Interaction potential of two strings ending on the same wall is considered. In the bulk these interactions are exponentially suppressed, while on the wall they are of a long range type due to the presence of massless fields in the world-volume theory on the wall. We find two configurations (see Fig. 5 and Fig. 7) for which the interaction potential disappears. They should correspond to 1/4-BPS solutions of the first-order equations (12).

Since the self-energy of the isolated boojum has a logarithmic part which makes it virtually impossible to detect a finite contribution discussed in [8] directly from the single-boojum configuration, we make an attempt to define it from the boojum-antiboojum configuration whose energy has no logarithmic parts. The outcome remains inconclusive, depending on which of the two conjectures formulated in Sect. 5 wins. The winner can be determined by a direct numerical calculation.

Acknowledgments

References

[1] E. R. C. Abraham and P. K. Townsend, Phys. Lett. B **291**, 85 (1992); Phys. Lett. B **295**, 225 (1992).

[2] G. R. Dvali and M. A. Shifman, Phys. Lett. B **396**, 64 (1997) (E) B **407**, 452 (1997) [hep-th/9612128].

[3] D. Tong, Phys. Rev. D **66**, 025013 (2002) [hep-th/0202012].

[4] M. Shifman and A. Yung, Phys. Rev. D **67**, 125007 (2003) [hep-th/0212293].

[5] M. Shifman and A. Yung, Phys. Rev. D **70**, 025013 (2004) [hep-th/0312257].

[6] K. Kakimoto and N. Sakai, Phys. Rev. D **68**, 065005 (2003) [hep-th/0306077].

[7] Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, Phys. Rev. Lett. **93**, 161601 (2004) [hep-th/0404198]; Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, Phys. Rev. D **70**, 125014 (2004) [hep-th/0405194]. M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, K. Ohta, N. Sakai and Y. Tachikawa, *Global structure of moduli space for BPS walls,* hep-th/0503033.

[8] N. Sakai and D. Tong, *Monopoles, vortices, domain walls and D-branes: The rules of interaction,* JHEP **0503**, 019 (2005) [arXiv:hep-th/0501207].

[9] A. Hanany and D. Tong, JHEP **0307**, 037 (2003) [hep-th/0306150].

[10] A. Marshakov and A. Yung, Nucl. Phys. B **647**, 3 (2002) [hep-th/0202172].

[11] R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi and A. Yung, Nucl. Phys. B **673**, 187 (2003) [hep-th/0307287].

[12] D. Tong, Phys. Rev. D **69** (2004) 065003, hep-th/0307302.

[13] M. Shifman and A. Yung, Phys. Rev. D **70**, 045004 (2004) [hep-th/0403149].

[14] A. Hanany and D. Tong, JHEP **0404**, 066 (2004) [hep-th/0403158].

[15] Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, Phys. Rev. D **71**, 065018 (2005) [arXiv:hep-th/0405129].

[16] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, arXiv:hep-th/0412048.

[17] R. Auzzi, S. Bolognesi, J. Evslin and K. Konishi, Nucl. Phys. B **686**, 119 (2004) [hep-th/0312233]; R. Auzzi, S. Bolognesi and J. Evslin, JHEP **0502** (2005) 046 [hep-th/0411074].

[18] E. B. Bogomol'nyi, *The stability of classical solutions*, Yad. Fiz. **24**, 861 (1976) [Sov. J. Nucl. Phys. **24**, 449 (1976), reprinted in *Solitons and Particles*, Eds. C. Rebbi and G. Soliani (World Scientific, Singapore, 1984), p. 389].

[19] A. I. Vainshtein and A. Yung, Nucl. Phys. B **614**, 3 (2001) [hep-th/0012250].

[20] A. Gorsky and M. A. Shifman, Phys. Rev. D **61**, 085001 (2000) [hep-th/9909015].

[21] C. G. Callan and J. M. Maldacena, Nucl. Phys. B **513**, 198 (1998) [hep-th/9708147].

[22] H. Oda, K. Ito, M. Naganuma and N. Sakai, Phys. Lett. B **471**, 140 (1999) [hep-th/9910095].

[23] M. A. Shifman and T. ter Veldhuis, Phys. Rev. D **62**, 065004 (2000) [hep-th/9912162].